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Existence of global solutions for a semilinear parabolic Cauchy problem

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1 Introduction

In this paper, we consider the following Cauchy problem

$$\begin{cases} w_t = \Delta w + |x|^l w^p, & x \in \mathbf{R}^n,\ t > 0, \\ w(x,0) = f(x), & x \in \mathbf{R}^n, \end{cases} \tag{1.1}$$

where $p > 1$, $l > -2$ and $n \geq 3$ are parameters, and f is a nonnegative bounded continuous function in $\mathbf{R}^n$. This problem is more general version of

$$\begin{cases} w_t = \Delta w + w^p, & x \in \mathbf{R}^n,\ t > 0, \\ w(x,0) = f(x), & x \in \mathbf{R}^n. \end{cases} \tag{1.2}$$

In 1966, Fijita [2] has proved that if $p < 1 + 2/n$, then the solution of (1.2) blows up in finite time for all $f \geq 0$ and $f \not\equiv 0$, and if $p > 1 + 2/n$, then (1.2) has a global classical solution when f satisfies $f(x) < \delta \exp(-|x|^2)$ where δ is sufficiently small positive number. Moreover, Lee and Ni [4] have shown that the global existence when the initial value f has polynomial decay near $x = \infty$. Their result is that if $p > 1 + 2/n$ and f satisfies $f(x) \sim (1 + |x|^2)^{-1/(p-1)}$, then (1.2) has a global classical solution and the solution $w(x,t)$ satisfies $||w(\cdot,t)||_{L^\infty(\mathbf{R}^n)} \sim t^{-1/(p-1)}$ as $t \to \infty$. Furthermore, for the problem (1.1), Wang [5] has derived the following result.

Theorem A. ([5]) *Suppose* $n \geq 3$, $l > -2$ *and* $p \geq (n+2+2l)/(n-2)$, *then there exists a small* $\mu > 0$ *such that if* $0 \leq f(x) \leq \mu(1+|x|)^{-(2+l)/2(p-1)}$ *in* $\mathbf{R}^n$, *then* (1.1) *has global solution* $w(x,t)$ *with* $\|w(\cdot,t)\|_{L^\infty(\mathbf{R}^n)} \leq Mt^{-(2+l)/2(p-1)}$.

In order to prove Theorem A, we introduce the following semilinear elliptic equation with a gradient term

$$\Delta u + \frac{1}{2}x \cdot \nabla u + \lambda u + |x|^l u^p = 0, \quad x \in \mathbf{R}^n. \tag{1.3}$$

where, n, l, p and λ are parameters. It is easily seen that $w(x,t)$ given by

$$w(x,t) = t^{-(2+l)/2(p-1)} u\left(x/\sqrt{t}\right) \tag{1.4}$$

satisfies the equation of (1.1) if and only if $u(x) : \mathbf{R}^n \to \mathbf{R}$ in (1.4) satisfies

$$\Delta u + \frac{1}{2}x \cdot \nabla u + \frac{2+l}{2(p-1)}u + |x|^l u^p = 0, \quad x \in \mathbf{R}^n. \tag{1.5}$$

Since we will consider the radial solutions ($u = u(r)$ with $r = |x|$) to (1.3), we need the following initial value problem

$$\begin{cases} u'' + \left(\dfrac{n-1}{r} + \dfrac{r}{2}\right)u' + \lambda u + r^l u^p = 0, & r > 0, \\ u(0) = \alpha > 0, \quad u'(0) = 0. \end{cases} \tag{1.6}$$

Note that when $l > -2$, (1.6) has a unique solution $u(r) \in C^1([0,\infty)) \cap C^2((0,\infty))$, which is denoted by $u(r;\alpha)$. Then Wang has shown the follwing result.

Theorem B. ([5]) *Suppose* $n \geq 3$, $l > -2$ *and* $p \geq (n+2+2l)/(n-2)$. *Then,*

(i) $\lim_{r\to\infty} r^{2\lambda}u(r;\alpha)$ *exists.*

(ii) *If* $\lim_{r\to\infty} r^{2\lambda}u(r;\alpha) = 0$, *then* $\lim_{r\to\infty} r^m u(r;\alpha) = 0$ *and* $\lim_{r\to\infty} r^m u_r(r;\alpha) = 0$ *for all positive* m.

(iii) *Let* $\lambda = (2+l)/2(p-1)$ *Then there exists some positive number* $\tilde{\alpha}$ *such that* $u(r;\tilde{\alpha}) > 0$ *for all* $r \geq 0$ *and* $u(r;\tilde{\alpha})$ *satisfies* $u(r;\tilde{\alpha}) \sim r^{-2\lambda}$ *as* $r \to \infty$.

Wang has shown Theorem A by using Theorem B. We introduce the sketch of the proof as follows:

Sketch of the proof of Theorem A. Let $\lambda = (2+l)/2(p-1)$. When the parameters n, l and p satisfy the assumptions, it follows from Theorem B that there exists some positive solution $u(r)$ with $u(r) \sim r^{-2\lambda}$ as $r \to \infty$ for the problem (1.6). Now, we set

$$\hat{w}(x,t) = (t+1)^{-\lambda} u\left(|x|/\sqrt{t+1}\right),$$

then $\hat{w}(x,t)$ is an upper solution of (1.1) if μ is sufficiently small positive number. Moreover, since the trivial solution can be a lower solution of (1.1), there exists a global solution of (1.1). ■

Namely, if we can show the existence of some positive solution $u(r)$ with $u(r;\alpha) \sim r^{-2\lambda}$ as $r \to \infty$ for the problem (1.6), then we can conclude the existence of a global solution to (1.1). Our first result is the following one.

Theorem 1.1 *Suppose $n \geq 3$. Then there exist some positive numbers α_0 and $l^* = l^*(n) \in (0,1)$ such that if $-2 < l < l^*$ and $1 + (2+l)/n < p < (n+2+2l)/(n-2)$, then*

(i) *For any $\alpha \in (0, \alpha_0)$, $u(r;\alpha) > 0$ for all $r \geq 0$ and $u(r;\alpha)$ satisfies $u(r;\alpha) \sim r^{-2\lambda}$ as $r \to \infty$.*

(ii) *$u(r;\alpha_0) > 0$ for all $r \geq 0$ and $u(r;\alpha_0)$ satisfies $\lim_{r\to\infty} r^{2\lambda}u(r;\alpha_0) = 0$.*

(iii) *For any $\alpha \in (\alpha_0, \infty)$, $u(\cdot,\alpha)$ has a zero in $(0,\infty)$.*

(*Especially, $l^*(1) = 2/3$ holds.*)

Moreover, the following theorem follows from Theorem1.1 and the sketch of the proof of Theorem A.

Theorem 1.2 *Suppose $n \geq 3$. Then there exists some positive number $l^* = l^*(n) \in (0,1)$ satisfies $-2 < l < l^*$ such that if $1+(2+l)/n < p < (n+2+2l)/(n-2)$, then there exists a small $\mu > 0$ which satisfies*

if $0 \leq f(x) \leq \mu(1+|x|)^{-(2+l)/2(p-1)}$ in $\mathbf{R}^n$, then (1.1) has global solution $w(x,t)$.

In order to prove Theorem 1.1, we apply the classification theorem by Yanagida and Yotsutani [7]. Let $\varphi(r)$ be a solution of

$$\begin{cases} \varphi'' + \left(\dfrac{n-1}{r} + 2r\right)\varphi' + \lambda\varphi = 0, \quad r > 0, \\ \varphi(0) = 1, \quad \varphi'(0) = 0. \end{cases} \tag{1.7}$$

For a solution $u(r)$ of (1.6), if we put

$$u(r) = \varphi(r)v(r), \tag{1.8}$$

then we see that $v(r)$ satisfies

$$\begin{cases} (g(r)v')' + g(r)K(r)|v|^{p-1}v = 0, \quad r > 0, \\ v(0) = \alpha > 0, \quad v'(0) = 0, \end{cases} \tag{1.9}$$

where

$$g(r) := r^{n-1}\exp\left(r^2/4\right)\varphi(r)^2, \quad K(r) := r^l|\varphi(r)|^{p-1}. \tag{1.10}$$

We should note that $\varphi(r) > 0$ on $[0,\infty)$ if $\lambda > -n$ by (i) of Proposition 2.1 in Section 2. To see whether $u(r)$ has a zero or not, we have only to check this property for $v(r)$. For this purpose, we employ the classification theorem by Yanagida and Yotsutani [7], which is stated as follows. Let $g(r)$ and $K(r)$ satisfy

$$\begin{cases} g(r) \in C^2([0,\infty)); \\ g(r) > 0 \quad \text{on} \quad (0,\infty); \\ 1/g(r) \notin L^1(0,1); \\ 1/g(r) \in L^1(1,\infty), \end{cases} \tag{g}$$

and

$$\begin{cases} K(r) \in C(0,\infty); \\ K(r) \geq 0 \quad \text{and} \quad K(r) \not\equiv 0 \quad \text{on} \quad (0,\infty); \\ h(r)K(r) \in L^1(0,1); \\ g(r)\left(h(r)/g(r)\right)^p K(r) \in L^1(1,\infty), \end{cases} \tag{K}$$

where

$$h(r) := g(r) \int_r^\infty g(s)^{-1} ds.$$

Moreover, define

$$G(r) := \frac{2}{p+1} g(r)h(r)K(r) - \int_0^r g(s)K(s)ds, \tag{1.11}$$

$$H(r) := \frac{2}{p+1} h(r)^2 \left(\frac{h(r)}{g(r)}\right)^p K(r) - \int_r^\infty h(s) \left(\frac{h(s)}{g(s)}\right)^p K(s)ds, \tag{1.12}$$

and

$$r_G := \inf\{r \in (0,\infty) : \ G(r) < 0\}, \quad r_H := \sup\{r \in (0,\infty) : \ H(r) < 0\}.$$

Theorem C. ([7]) *Assume that $g(r)$ and $K(r)$ satisfy the conditions (g) and (K). Let $v(r;\alpha)$ be a solution of*

$$\begin{cases} (g(r)v')' + g(r)K(r)(v^+)^p = 0, \quad r > 0, \\ v(0) = \alpha > 0, \quad v'(0) = 0, \end{cases} \tag{1.13}$$

where $v^+ := \max\{v, 0\}$, and suppose that $G(r) \not\equiv 0$ on $(0,\infty)$.

(i) *If*

$$0 < r_H \le r_G < \infty, \tag{1.14}$$

then there exists a unique positive number α_0 such that the structure of solutions to (1.13) is as follows.

(a) *For every $\alpha \in (\alpha_0, \infty)$, $v(r;\alpha)$ has a zero in $(0,\infty)$.*

(b) *If $\alpha = \alpha_0$, then $v(r;\alpha) > 0$ on $[0,\infty)$ and*

$$0 < \lim_{r\to\infty} \left(\int_r^\infty g(s)^{-1} ds\right)^{-1} v(r;\alpha) < \infty. \tag{1.15}$$

(c) *For every $\alpha \in (0,\alpha_0)$, $v(r;\alpha) > 0$ on $[0,\infty)$ and*

$$\lim_{r\to\infty} \left(\int_r^\infty g(s)^{-1} ds\right)^{-1} v(r;\alpha) = \infty. \tag{1.16}$$

(ii) *If $r_G < \infty$ and $r_H = 0$ (i.e., $H(r) \ge 0$ on $[0,\infty)$), then $v(r;\alpha)$ is positive on $[0,\infty)$ and satisfies (1.16) for every $\alpha > 0$.*

(iii) *If $r_G = \infty$ (i.e., $G(r) \ge 0$ on $[0,\infty)$), then $v(r;\alpha)$ has a zero in $(0,\infty)$ for every $\alpha > 0$.*

2 Proof of Theorem 1.1

In order to prove Theorem 1.1, we prepare the following proposition which is shown in [1] and [3].

Proposition 2.1 *Let φ be the solution of* (1.7) *and suppose* $0 < \lambda < n/2$. *Then,*

(i) $\varphi(r) > 0$ *and* $\varphi'(r) < 0$ *in* $(0, \infty)$.

(ii) $\lim_{r \to \infty} r^{2\lambda}\varphi(r)$ *exists and positive.*

(iii) $\varphi(r) = 1 - \dfrac{\lambda}{2n} r^2 + o(r^2)$ *as* $r \to \infty$.

(iv) $\exp(r^2/4)\varphi(r)$ *is an increasing function of* $r \in [0, \infty)$.

(v) *If* $0 < \lambda \leq (n-2)/2$ *and*$(n-2)/2 < \lambda < n/2$, *then*

$$-2\lambda < \frac{r\varphi'(r)}{\varphi(r)} < 0 \tag{2.1}$$

and

$$-\frac{4\lambda}{n - 2\lambda} < \frac{r\varphi'(r)}{\varphi(r)} < 0 \tag{2.2}$$

for all $r \in (0, \infty)$, *respectively.*

(vi) *Set*

$$m(\lambda) := \begin{cases} 2\lambda & \text{if} \quad 0 < \lambda \leq \dfrac{n-2}{2}, \\ \dfrac{4\lambda}{n - 2\lambda} & \text{if} \quad \dfrac{n-2}{2} < \lambda < \dfrac{n}{2}. \end{cases}$$

Then for each $\lambda \in (0, n/2)$, $r^{m(\lambda)}\varphi(r)$ *is an increasing function of* $r \in [0, \infty)$.

By using Proposition 2.1, we can check the conditions imposed on the coefficients of equation of (1.9).

Lemma 2.1 *If* $n \geq 3$ *and* $0 < \lambda < n/2$, *then* $g(r) := r^{n-1} \exp(r^2/4)\, \varphi(r)^2$ *and* $K(r) := r^l |\varphi(r)|^{p-1}$ *satisfy* (g) *and* (K), *respectively.*

Therefore, $g(r)$ and $K(r)$ are admissible. Substituting their definition (1.10) into $G(r)$ and $H(r)$, we obtain

$$\begin{aligned} G(r) &:= \frac{2}{p+1}g(r)h(r)K(r) - \int_0^r g(s)K(s)ds \\ &= \frac{2}{p+1}r^{2n-2+l}\exp\left(\frac{r^2}{2}\right)\varphi(r)^{p+3}\left\{\int_r^\infty s^{1-n}\exp\left(-\frac{s^2}{4}\right)\varphi(s)^{-2}ds\right\} \\ &\qquad - \int_0^r s^{n-1+l}\exp\left(\frac{s^2}{4}\right)\varphi(s)^{p+1}ds, \end{aligned}$$

$$\begin{aligned} H(r) &:= \frac{2}{p+1}h(r)^2\left(\frac{h(r)}{g(r)}\right)^p K(r) - \int_r^\infty h(s)\left(\frac{h(s)}{g(s)}\right)^p K(s)ds \\ &= \frac{2}{p+1}r^{2n-2+l}\exp\left(\frac{r^2}{2}\right)\varphi(r)^{p+3}\left\{\int_r^\infty s^{1-n}\exp\left(-\frac{s^2}{4}\right)\varphi(s)^{-2}ds\right\}^{p+2} \\ &\qquad - \int_r^\infty s^{n-1+l}\exp\left(\frac{s^2}{4}\right)\varphi(s)^{p+1}\left\{\int_s^\infty t^{1-n}\exp\left(-\frac{t^2}{4}\right)\varphi(t)^{-2}dt\right\}^{p+1} ds. \end{aligned}$$

In order to prove Theorem 1.1, we must show condition (1.14). To do so, we will investigate the profiles of $G(r)$ and $H(r)$. First, we will study the increase and decrease. Differentiation yields

$$G'(r) = \left(\int_r^\infty g(s)^{-1}ds\right)^{-p-1} H'(r) = \frac{2}{p+1}g(r)K(r)\left(\Phi(r) - \frac{p+3}{2}\right), \tag{2.3}$$

where

$$\begin{aligned} \Phi(r) &:= \left(2g'(r) + \frac{g(r)K'(r)}{K(r)}\right)\int_r^\infty g(s)^{-1}ds \\ &= r^{n-2}\exp\left(\frac{r^2}{4}\right)\varphi(r)^2\left[r^2 + \{2(n-1)+l\} + (p+3)\left(\frac{r\varphi'(r)}{\varphi(r)}\right)\right] \\ &\qquad \times \int_r^\infty s^{1-n}\exp\left(-\frac{s^2}{4}\right)\varphi(s)^{-2}ds. \end{aligned} \tag{2.4}$$

In view of (2.3), $G(r)$ and $H(r)$ have the same extremal points, namely those $r > 0$ which satisfy $\Phi(r) = (p+3)/2$. So in order to know the sign of $G'(r)$ and $H'(r)$, we must study the relation between $\Phi(r)$ and $(p+3)/2$. We first consider the behaviour of $\Phi(r)$ near $r = 0$ and $r = \infty$.

Lemma 2.2 *Suppose $n \geq 3$ and $0 < \lambda < n/2$. Then*

$$\lim_{r\to 0}\Phi(r) = \frac{2n-2+l}{n-2} \quad \textit{and} \quad \lim_{r\to\infty}\Phi(r) = 2.$$

Proof. Using l'Hospital's theorem, in view of the definition of $\Phi(r)$ we derive

$$\begin{aligned}\lim_{r\to\infty}\Phi(r) &= \lim_{r\to\infty}\frac{\frac{d}{dr}\int_r^\infty s^{1-n}\exp(-s^2/4)\varphi(s)^{-2}ds}{\frac{d}{dr}\left[r^{n-2}\exp(r^2/4)\varphi(r)^2\left\{r^2+2(n-1)+l+(p+3)\left(\frac{r\varphi'}{\varphi}\right)\right\}\right]^{-1}}\\ &= 2\end{aligned}$$

with noting (2.1) and (2.2). The case $r\to 0$ is done similarly. ∎

Note that $\Phi(r)$ is continuous in $[0,\infty)$ and $(p+3)/2$ satisfies

$$(2<)\ 2+\frac{2+l}{2n}<\frac{p+3}{2}<\frac{2n-2+l}{n-2} \tag{2.5}$$

if and only if $1+\dfrac{2+l}{n}<p<\dfrac{n+2+2l}{n-2}$. And we can get the following lemma which will be proved in the following Section 3.

Lemma 2.3 *Under the assumptions on n, l, p and λ in Theorem 1.1, there exists a unique number $r^*\in(0,\infty)$ satisfying $\Phi(r^*)=(p+3)/2$ and*

$$\begin{cases}\Phi(r)>\dfrac{p+3}{2} & \text{in}\quad [0,r^*),\\[2mm] \Phi(r)\le\dfrac{p+3}{2} & \text{in}\quad (r^*,\infty).\end{cases} \tag{2.6}$$

Therefore, it follows from (2.3) and Lemma 2.3 that

Lemma 2.4 *Under the assumptions of Theorem 1.1, there exists a unique number $r^*\in(0,\infty)$ such that $G(r)$ and $H(r)$ are increasing in $[0,r^*)$ and decreasing in (r^*,∞).*

Moreover, in order to locate r_G and r_H, we need to determine the behaviour of $G(r)$ and $H(r)$ near $r=0$ and $r=\infty$.

Lemma 2.5 *Under the assumptions of Theorem 1.1,*

(i) $\lim_{r\to\infty}G(r)=-\infty$.

(ii) $\lim_{r\to 0}G(r)=0$.

(iii) $\liminf_{r\to\infty}H(r)\ge 0$.

(iv) $\limsup_{r\to 0}H(r)<0$.

We can show this lemma by using Proposition 2.1. Now we can prove Theorem 1.1.

Proof of Theorem 1.1. First of all, we must note that

$$0<\lambda<\frac{n}{2} \iff 1+\frac{2+l}{n}<p<\infty$$

holds when $\lambda=(2+l)/2(p-1)$. As is already seen in Lemma 2.4, both $G(r)$ and $H(r)$ have exactly one local maximum at $r^*\in(0,\infty)$. Moreover, in view of Lemma 2.5 $H(r)$ is negative near $r=0$ and positive for large r. Thus $H(r^*)>0$ and $0<r_H<r^*$. Besides, we obtain $G(r^*)>0$ from $G(0)=0$, and the negativity of $G(r)$ for large r yields $0<r^*<r_G<\infty$; so we conclude that condition (1.14) holds. Thus from Theorem C there exists a unique positive number α_0 such that for every $\alpha\in(\alpha_0,\infty)$ $v(\cdot;\alpha)$, i.e., $u(\cdot;\alpha)$ has a zero in $(0,\infty)$. Moreover, for every $\alpha\in(0,\alpha_0]$ $v(\cdot;\alpha)$ is positive in $[0,\infty)$ and

$$\lim_{r\to\infty}\left\{\int_r^\infty s^{1-n}\exp\left(-\frac{s^2}{4}\right)\varphi(s)^{-2}ds\right\}^{-1}v(r;\alpha)\begin{cases}\in(0,\infty) & \text{if } \alpha=\alpha_0,\\ =\infty & \text{if } \alpha\in(0,\alpha_0).\end{cases}$$

Integrating by parts, we obtain

$$\int_r^\infty s^{1-n}\exp\left(-\frac{s^2}{4}\right)\varphi(s)^{-2}ds=2r^{-n}\exp\left(-\frac{r^2}{4}\right)\varphi(r)^{-2}(1+o(1))$$

as $r\to\infty$. (Here we use the boundedness of $r\varphi'/\varphi$.) Taking the decay rate of $\varphi(r)$ and the definition of $v(r)$ into account (see Proposition 2.1 (ii)), this estimate immediately shows that (1.15) with $g(r)=r^{n-1}\exp(r^2/4)\varphi^2$ is equivalent to $\lim_{r\to\infty}r^{2\lambda}u(r)=0$. At the same time, we obtain that (1.16) with $g(r)=r^{n-1}\exp(r^2/4)\varphi^2$ is equivalent to $\lim_{r\to\infty}r^{2\lambda}u(r)>0$, i.e. $u(r;\alpha)$ satisfies $u(r;\alpha)\sim r^{-2\lambda}$ as $r\to\infty$ for every $\alpha\in(0,\alpha_0)$. ∎

3 Proof of Lemma 2.3

We will show there is exactly one crossing point of $q=\Phi(r)$ and $q=(p+3)/2$ in (r,q)-plane. Our strategy is to investigate the sign of $\Phi'(r_*)$ for r_* satisfying $\Phi(r_*)=(p+3)/2$. Here the existence of r_* is guaranteed by Lemma 2.2, the continuity of $\Phi(r)$ and (2.5).

Define

$$\begin{cases} \Omega_1 := \{r_* \in (0,\infty); \Phi'(r_*) > 0\}, \\ \Omega_2 := \{r_* \in (0,\infty); \Phi'(r_*) = 0\}, \\ \Omega_3 := \{r_* \in (0,\infty); \Phi'(r_*) < 0\}. \end{cases}$$

Then we obtain the following result.

Lemma 3.1 *Suppose the assumptions on n, l, p and λ in Theorem 1.1. Then*

(i) Ω_1 *is empty.*

(ii) Ω_2 *consists of at most one element.*

(iii) Ω_3 *consists of at most one element.*

Proof. Differentiating $\Phi(r)$, we get

$$\begin{aligned}\Phi'(r) = \Bigg[&\frac{r^4}{2} + \{(2n-1) - \lambda(p+3)\}r^2 + 2(n-1)(n-2) \\ &+2\{r^2 + 2(n-1) + l\}\left(\frac{r\varphi'}{\varphi}\right) + (p+3)\left(\frac{r\varphi'}{\varphi}\right)^2\Bigg] \\ &\times r^{n-3}\exp\left(\frac{r^2}{4}\right)\varphi(r)^2\int_r^\infty s^{1-n}\exp\left(-\frac{s^2}{4}\right)\varphi(s)^{-2}ds \\ &-\frac{1}{r}\left\{r^2 + 2(n-1) + l + (p+3)\left(\frac{r\varphi'}{\varphi}\right)\right\}.\end{aligned} \tag{3.1}$$

For any r_* satisfying $\Phi(r_*) = (p+3)/2$ the equality

$$\begin{aligned}&r_*^{n-3}\exp\left(\frac{r_*^2}{4}\right)\varphi(r_*)^2\int_{r_*}^\infty s^{1-n}\exp\left(-\frac{s^2}{4}\right)\varphi(s)^{-2}ds \\ &= \frac{p+3}{2r_*\left\{r_*^2 + 2(n-1) + l + (p+3)\left(\frac{r_*\varphi'(r_*)}{\varphi(r_*)}\right)\right\}}\end{aligned} \tag{3.2}$$

holds. Note that

$$2r_*\left\{r_*^2 + 2(n-1) + l + (p+3)\left(\frac{r_*\varphi'(r_*)}{\varphi(r_*)}\right)\right\} > 0, \tag{3.3}$$

since the left-hand side of (3.2) is positive. Combining (3.1) with $r = r_*$ and (3.2), we obtain

$$\Phi'(r_*) = \frac{\Psi(r_*)}{2r_*\left\{r_*^2 + 2(n-1) + l + (p+3)\left(\frac{r_*\varphi'(r_*)}{\varphi(r_*)}\right)\right\}},$$

where

$$\begin{aligned}\Psi(r) := {} & \frac{p-1}{2}r^4 + \left\{(2n-1)p - 2n + 5 - \lambda(p+3)^2 + \frac{p-5}{2}l\right\}r^2 \\ & + \{2(n-1)+l\}\{(n-2)p - (n+2) - 2l\} \\ & - 2(p+3)\{r^2 + 2(n-1) + l\}\left(\frac{r\varphi'}{\varphi}\right) - (p+3)^2\left(\frac{r\varphi'}{\varphi}\right)^2.\end{aligned} \tag{3.4}$$

In view of (3.3), we will investigate the sign of $\Psi(r_*)$ instead of $\Phi'(r_*)$. Using (2.1) and (2.2), it is easily seen that

$$\begin{cases} \lim\limits_{r\to\infty} \Psi(r) = +\infty, \\ \lim\limits_{r\to 0} \Psi(r) = \{2(n-1)+l\}\{(n-2)p - (n+2) - 2l\} < 0. \end{cases} \tag{3.5}$$

Moreover, we have the following lemma whose proof will be given at the end of this section.

Lemma 3.2 *Under the assumptions on n, l, p and λ in Theorem 1.1, there exists a unique number $\hat{r} \in (0,\infty)$ satisfying $\Psi(\hat{r}) = 0$ such that*

$$\Psi(r) < 0 \quad in \quad [0,\hat{r}) \quad and \quad \Psi(r) > 0 \quad in \quad (\hat{r},\infty).$$

Recalling that the sign of $\Phi'(r_*)$ is equivalent to that of $\Psi(r_*)$, from Lemma 3.2 we get the sign of $\Phi'(r_*)$ as follows:

$$\begin{cases} \Phi'(r_*) < 0 & \text{if} \quad r_* \in [0,\hat{r}), \\ \Phi'(r_*) = 0 & \text{if} \quad r_* = \hat{r}, \\ \Phi'(r_*) > 0 & \text{if} \quad r_* \in (\hat{r},\infty). \end{cases} \tag{3.6}$$

First, we will show (i). If $q = \Phi(r)$ and $q = (p+3)/2$ cross in $(\hat{r},\infty)$, then there exists a unique number r'_* in $(\hat{r},\infty)$ such that $\Phi(r'_*) = (p+3)/2$ with $\Phi'(r'_*) > 0$. But it is impossible because of $\Phi(r) \to 2$ as $r \to \infty$. Therefore, $\Omega_1 = \emptyset$.

Moreover, if $q = \Phi(r)$ and $q = (p+3)/2$ cross in $[0,\hat{r})$, then there exists a unique number $r''_* \in [0,\hat{r})$ such that $\Phi(r''_*) = (p+3)/2$ with $\Phi'(r''_*) < 0$. Therefore, $\Omega_3 = \emptyset$ or

$\Omega_3 = \{r_*''\}$; so we conclude (iii). Statement (ii) is trivial in view of (3.6). ∎

Thus we conclude that the relation between $q = \Phi(r)$ and $q = (p+3)/2$ in (r,q)-plane is one of the following:

(a) $\Phi(r) > (p+3)/2$ in $[0, r_*'')$ and $\Phi(r) < (p+3)/2$ in (r_*'', ∞),

(b) $\Phi(r) > (p+3)/2$ in $[0, \hat{r})$ and $\Phi(r) < (p+3)/2$ in $(\hat{r}, \infty)$,

(c) $\Phi(r) > (p+3)/2$ in $[0, r_*'')$ and $\Phi(r) < (p+3)/2$ in $(r_*'', \infty)\backslash\hat{r}$.

(Here we used the notation introduced in the proof of Lemma 3.1.) Therefore, by putting $r^* := r_*''$ in cases (a) and (c) or $r^* = \hat{r}$ in case (b), (i) of Lemma 2.3 holds.

Proof of Lemma 3.2. In order to prove Lemma 3.2, we will investigate the sign of $\Psi'(\tilde{r})$ for $\tilde{r}$ satisfying $\Psi(\tilde{r}) = 0$. If we set

$$\begin{cases} X(r) := r\varphi'/\varphi; \\ A := -(p+3)^2; \\ B(r) := -2(p+3)\{r^2 + 2(n-1) + l\}; \\ C(r) := \dfrac{p-1}{2}r^4 + \left\{(2n-1)p - 2n + 5 - \lambda(p+3)^2 + \dfrac{l}{2}(p-5)\right\}r^2 \\ \qquad\qquad + \{2(n-1)+l\}\{(n-2)p - (n+2) - 2l\}, \end{cases}$$

then $\Psi(r)$ can be rewritten as

$$\Psi(r) \equiv AX(r)^2 + B(r)X(r) + C(r). \tag{3.7}$$

From (1.7), we can express $X'(r)$ in terms of $X(r)$ as

$$X'(r) = -\frac{1}{r}X(r)^2 - \frac{n-2}{r}X(r) - \frac{r}{2}X(r) - \lambda r. \tag{3.8}$$

So differentiating (3.7) and using (3.8), we obtain

$$\begin{aligned} \Psi'(r) = -\frac{2A}{r}X(r)^3 - \left\{\frac{2A(n-2)+B(r)}{r} + Ar\right\}X(r)^2 - \left\{\frac{(n-2)B(r)}{r}\right. \\ \left. -B'(r) + \left(2\lambda A + \frac{B(r)}{2}\right)r\right\}X(r) - \lambda B(r)r + C'(r). \end{aligned} \tag{3.9}$$

From $\Psi(\tilde{r}) = 0$, $X(\tilde{r})^2$ and $X(\tilde{r})^3$ can be replaced by

$$\begin{cases} X(\tilde{r})^2 = -\dfrac{B(\tilde{r})}{A}X(\tilde{r}) - \dfrac{C(\tilde{r})}{A}, \\ X(\tilde{r})^3 = \left(\dfrac{B(\tilde{r})^2}{A^2} - \dfrac{C(\tilde{r})}{A}\right)X(\tilde{r}) + \dfrac{B(\tilde{r})C(\tilde{r})}{A^2}. \end{cases} \tag{3.10}$$

Substituting (3.10) into (3.9), all the terms containing $X(\tilde{r})$ vanish and $\tilde{r}\Psi'(\tilde{r})$ turns out to be a polynomial in $\tilde{r}^2$ of degree 3:

$$\begin{aligned} (p+3)\tilde{r}\Psi'(\tilde{r}) \;\; = \;\; & \frac{(p+1)(p-1)}{2}\tilde{r}^6 + \eta(l,p,n,\lambda)\tilde{r}^4 + \kappa(l,p,n,\lambda)\tilde{r}^2 \\ & + 2\{2(n-1)+l\}\{(n-2)p+n-4-l\}\{(n-2)p-(n+2)-2l\} \end{aligned}$$

where

$$\eta(l,p,n,\lambda) := -\lambda p^3 + \left(3n - 5\lambda - 1 + \frac{l}{2}\right)p^2 - 3(\lambda - 2 + l)p - 3\left(n - 3\lambda - 1 + \frac{l}{2}\right),$$

and

$$\begin{aligned} \kappa(l,p,n,\lambda) \;\; := \;\; & -3(p-1)l^2 + \left[(2n-3)p^2 - 6(2n-1)p - 3(2n+1) + 4\lambda(p+3)^2\right] l \\ & + 2(n-1)(p-1)\{-\lambda p^2 + 3(n-2\lambda-1)p + 3(n-3\lambda-1)\}. \end{aligned}$$

We want to determine the sign of $\Psi'(\tilde{r})$. Setting $x = \tilde{r}^2$, we have $(p+3)\tilde{r}\Psi'(\tilde{r}) = \Theta(x)$, where $\Theta(x)$ is a polynomial of degree 3 given by

$$\begin{aligned} \Theta(x) \;\; := \;\; & \frac{(p+1)(p-1)}{2}x^3 + \eta(p,n,\lambda)x^2 + \kappa(p,n,\lambda)x \\ & + 2\{2(n-1)+l\}\{(n-2)p+n-4-l\}\{(n-2)p-(n+2)-2l\} \end{aligned}$$

for $x \in [0,\infty)$. Concerning the profile of $\Theta(x)$, we readily see

$$\begin{cases} \Theta(0) = 2\{2(n-1)+l\}\{(n-2)p+n-4-l\}\{(n-2)p-(n+2)-2l\}, \\ \lim_{x\to\infty}\Theta(x) = +\infty. \end{cases} \tag{3.11}$$

(Here, we must note that $(n-2)p+n-4-l > 0$ holds if $n \geq 3$, $p > 1 + (2+l)/n$ and $-2 < l < n^2 - 2n - 2$.) What may happen in $(0,\infty)$ is clarified by the following lemmas.

Lemma 3.3 *Suppose* $n \geq 3$, $-2 < l < 1$, $1 + (2+l)/n < p < (n+2+2l)/(n-2)$ *and set*

$$\lambda^*(l) := \frac{(n-2)\{l^2 + 2(n+4)l + 4(2n+1)\}}{4(l+2)\{l+2(n-1)\}}.$$

Then $\Theta(x)$ has exactly one zero in $(0,\infty)$ if $0<\lambda<\lambda^*(l)$.

Proof. Note that it follows from $n^2-2n-2\geq 1$ for any $n\geq 3$ that $\Theta(0)<0$ under the assumptions on n, l and p. Differentiating $\Theta(x)$, we obtain

$$\Theta'(x)=\frac{3}{2}(p+1)(p-1)x^2+2\eta(l,p,n,\lambda)x+\kappa(l,p,n,\lambda).$$

Now we will show $\eta(l,p,n,\lambda)$ is positive under the assumptions on n, l, p and λ. In fact, define

$$f_1(p):=\eta(l,p,n,\lambda),\qquad p\in\left(1,\frac{n+2+2l}{n-2}\right)$$

then we observe

$$f_1(1)=8-4l>0 \tag{3.12}$$

and

$$\begin{aligned}(n-2)^3 f_1&\left(\frac{n+2+2l}{n-2}\right)\\ &=2\{l+2(n-1)\}[(n-2)\{l^2+2(n+4)l+4(2n+1)\}\\ &\quad -4(l+2)\{l+2(n-1)\}\lambda]>0.\end{aligned} \tag{3.13}$$

From (3.12) and (3.13), it is sufficient to show that $f_1(p)$ does not have any local minimum in $(1,(n+2+2l)/(n-2))$. So consider the sign of $f_1''(\tilde{p})$, where $\tilde{p}$ satisfies $f_1'(\tilde{p})=0$. Using

$$f_1'(p)=-3\lambda p^2+2\left(3n-5\lambda-1+\frac{l}{2}\right)p-3(\lambda-2+l)$$

and

$$f_1''(p)=-6\lambda p+2\left(3n-5\lambda-1+\frac{l}{2}\right),$$

we obtain

$$\begin{aligned}\tilde{p}f_1''(\tilde{p})&=-6\lambda\tilde{p}^2+2\left(3n-5\lambda-1+\frac{l}{2}\right)\tilde{p}\\ &=-3\lambda(\tilde{p}^2-1)+3(l-2)<0.\end{aligned}$$

Thus, if $f_1(p)$ has an extremum, then it must be a local maximum and we conclude

$$\eta(l,p,n,\lambda)=f_1(p)\geq\min\left\{f_1(1),f_1\left(\frac{n+2+2l}{n-2}\right)\right\}>0.$$

Since $\eta(l,p,n,\lambda) > 0$, the quadratic equation $\Theta'(x) = 0$ does not have two solutions in $(0,\infty)$; so $\Theta(x)$ has at most one extremum. Therefore, either $\Theta(x)$ is increasing in $[0,\infty)$, or $\Theta(x)$ has a local minimum at $\hat{x}_1 \in (0,\infty)$ such that $\Theta(x)$ is decreasing in $[0,\hat{x}_1)$ and increasing in $(\hat{x}_1,\infty)$. Thus from (3.11) and $\Theta(0) < 0$ we can conclude that $\Theta(x)$ has exactly one zero in $(0,\infty)$. ■

Here, note that $\lim_{l\to -2+0} \lambda^*(l) = +\infty$ and the following result which is easily shown.

Lemma 3.4 *$\lambda^*(l)$ is decreasing in $(-2,\infty)$ and $\lim_{l\to\infty} \lambda^*(l) = \dfrac{n-2}{4} \left(< \dfrac{n}{2}\right)$.*

So for $l \in (-2,1)$, which satisfies $\lambda^*(l) < n/2$, and $\lambda \in (\lambda^*(l), n/2)$, the profile of $\Theta(x)$ has not been derived. First, we will prove the following result.

Lemma 3.5 *Let $n = 3$. If $-2 < l < 2/3$, then $\Theta(x)$ has exactly one zero in $(0,\infty)$ for any $\lambda \in (0,3/2)$ and $p \in (1+(2+l)/3, 5+2l)$.*

Proof. Since $\lambda^*(2/3) = 85/112 > 3/4$, for any $\lambda \in (0,3/4)$, $\Theta(x)$ has exactly one zero in $(0,\infty)$ from Lemmas 3.3 and 3.4. So we suppose $3/4 \le \lambda < 3/2$ and set

$$\begin{aligned} f_2(l) &:= \kappa(l,p,3,\lambda) \\ &= -3(p-1)l^2 + \left\{3(p^2-10p-7) + 4\lambda(p+3)^2\right\} l \\ &\quad + 4(p-1)\left\{-\lambda p^2 + 6(1-\lambda)p + 3(2-3\lambda)\right\}. \end{aligned}$$

Then we get

$$\begin{aligned} f_2(l) &= -3(p-1)\left\{l - \frac{3(p^2-10p-7)+4\lambda(p+3)^2}{6(p-1)}\right\}^2 \\ &\quad + \frac{\{3(p^2-10p-7)+4\lambda(p+3)^2\}^2}{12(p-1)} \\ &\quad + 4(p-1)\left\{-\lambda p^2 + 6(1-\lambda)p + 3(2-3\lambda)\right\} \end{aligned}$$

and

$$\begin{aligned} \frac{3(p^2-10p-7)+4\lambda(p+3)^2}{6(p-1)} &\ge \frac{3(p^2-10p-7)+4\cdot\frac{3}{4}(p+3)^2}{6(p-1)} \\ &= \frac{6(p-1)^2}{6(p-1)} \end{aligned}$$

$$
\begin{aligned}
&= p-1 \\
&> 1+\frac{2+l}{3}-1 \\
&= \frac{2+l}{3}.
\end{aligned}
$$

Thus the axis of quadratic function $f_2(l)$ is positive for all $l > -2$. Moreover, since $f_2(0) < 0$ under the assumptions on p and λ with $l = 0$ (see Lemma 3.2 in [3]), we obtain $f_2(l)$ is negative in $(-2, 0]$. Furthermore, if $0 < l < 2/3$, then $p > 5/3$, the axis of $f_2(l)$ is greater than $2/3$, and

$$
\begin{aligned}
\frac{3}{2} f_2\left(\frac{2}{3}\right) &= -6\lambda p^3 + (-26\lambda + 39)p^2 + (6\lambda - 32)p + 90\lambda - 55 \\
&= -(3p-5)\left\{2\lambda(p+3)^2 - 13p - 11\right\} \\
&< -(3p-5)\left\{2 \cdot \frac{3}{4}(p+3)^2 - 13p - 11\right\} \\
&= -(3p-5)\left\{\frac{3}{2}\left(p - \frac{4}{3}\right)^2 - \frac{1}{6}\right\} \\
&< -(3p-5)\left\{\frac{3}{2}\left(\frac{5}{3} - \frac{4}{3}\right)^2 - \frac{1}{6}\right\} \\
&= 0
\end{aligned}
$$

holds. Therefore, we have $f_2(l)$ is also negative in $(0, 2/3)$. Hence, we have $f_2(l) < 0$, i.e., $\kappa(l, p, 3, \lambda) < 0$ under the assummptions on l, p and λ. Since $\kappa(l, p, 3, \lambda) < 0$, the quadratic equation $\Theta'(x) = 0$ has exactly one solution in $(0, \infty)$; so $\Theta(x)$ has a local minimum at $\hat{x}_2 \in (0, \infty)$ such that $\Theta(x)$ is decreasing in $[0, \hat{x}_2)$ and increasing in $(\hat{x}_2, \infty)$. Thus it follows from (3.11) that $\Theta(x)$ has a unique zero in $(0, \infty)$. ∎

Next, we will show the following lemma.

Lemma 3.6 *Let $n \geq 4$. Then there exists some positive number $l^* = l^*(n) \in (0, 1)$ such that if $-2 < l < l^*$ and $1 + (2+l)/n < p < (n+2+2l)/(n-2)$, $\Theta(x)$ has exactly one zero in $(0, \infty)$ for any $\lambda \in (0, n/2)$.*

Proof. Since it is easily seen

$$
\lambda^*(1) = \frac{(n-2)(10n+13)}{12(2n-1)}
$$

and

$$\frac{(n-2)(10n+13)}{12(2n-1)} > \frac{3(n-1)}{8} \quad \text{for} \quad n \geq 4,$$

for any $\lambda \in (0, 3(n-1)/8)$, $\Theta(x)$ has exactly one zero in $(0, \infty)$ from Lemmas 3.3 and 3.4. So we suppose $3(n-1)/8 \leq \lambda < n/2$ and set $f_3(l) := \kappa(l, p, n, \lambda)$. Then we get

$$\begin{aligned} f_3(l) &= -3(p-1)\left\{l - \frac{(2n-3)p^2 - 6(2n-1)p - 3(2n+1) + 4\lambda(p+3)^2}{6(p-1)}\right\}^2 \\ &\quad + \frac{\{(2n-3)p^2 - 6(2n-1)p - 3(2n+1) + 4\lambda(p+3)^2\}^2}{12(p-1)} \\ &\quad + 2(n-1)(p-1)\{-\lambda p^2 + 3(n-2\lambda-1)p + 3(n-3\lambda-1)\} \end{aligned}$$

and

$$\begin{aligned} &\frac{(2n-3)p^2 - 6(2n-1)p - 3(2n+1) + 4\lambda(p+3)^2}{6(p-1)} \\ &\geq \frac{(2n-3)p^2 - 6(2n-1)p - 3(2n+1) + 4 \cdot \frac{3(n-1)}{8}(p+3)^2}{6(p-1)} \\ &= \frac{(7n-9)p^2 - 6(n+1)p + 3(5n-11)}{12(p-1)} \\ &= \frac{1}{12}\left\{(7n-9)(p-1) + 8(n-3) + \frac{16(n-3)}{p-1}\right\}. \end{aligned}$$

Thus the axis of $f_3(l)$ is positive for all $l > -2$. Moreover, since $f_3(0) < 0$ under the assumptions on n, p and λ with $l = 0$ (see Lemma 3.2 in [3]), we obtain $f_3(l)$ is negative in $(-2, 0]$. Furthermore, if $n \geq 4$ and $0 < l < 1$, then $1 < 1 + 2/n < p < \frac{n+4}{n-2} < 4$, the axis of $f_3(l)$ is greater than 1. In fact,

$$\begin{aligned} &\frac{(2n-3)p^2 - 6(2n-1)p - 3(2n+1) + 4\lambda(p+3)^2}{6(p-1)} \\ &\geq \frac{1}{12}\left\{(7n-9)(p-1) + 8(n-3) + \frac{16(n-3)}{p-1}\right\} \\ &> \frac{1}{12}\left\{8(n-3) + \frac{16(n-3)}{4-1}\right\} \\ &= \frac{10}{9}(n-3) > 1 \end{aligned}$$

holds. Therefore, there exists some positive number $l^* = l^*(n) \in (0, 1)$ such that $f_3(l)$ is negative in $(0, l^*)$. Hence, we have $f_3(l) < 0$, i.e., $\kappa(l, p, n, \lambda) < 0$ under the assummptions on n, l, p and λ. Since $\kappa(l, p, n, \lambda) < 0$, the quadratic equation $\Theta'(x) = 0$ has exactly one solution in $(0,\infty)$; so $\Theta(x)$ has a local minimum at $\hat{x}_3 \in (0,\infty)$ such that $\Theta(x)$ is decreasing in $[0,\hat{x}_3)$ and increasing in $(\hat{x}_3,\infty)$. Thus it follows from (3.11) that $\Theta(x)$ has a unique zero in $(0,\infty)$. ■

Thus in view of Lemmas 3.3, 3.5, 3.6 and (3.11), there exists a unique number $x_0 \in (0,\infty)$ such that

$$\begin{cases} \Theta(x) < 0 \quad \text{in} \quad (0,x_0), & \text{i.e.} \quad \Psi'(\tilde{r}) < 0 \quad \text{if} \quad \tilde{r} \in (0,\sqrt{x_0}), \\ \Theta(x_0) = 0, & \text{i.e.} \quad \Psi'(\tilde{r}) = 0 \quad \text{if} \quad \tilde{r} = \sqrt{x_0}, \\ \Theta(x) > 0 \quad \text{in} \quad (x_0,\infty), & \text{i.e.} \quad \Psi'(\tilde{r}) > 0 \quad \text{if} \quad \tilde{r} \in (\sqrt{x_0},\infty). \end{cases}$$

Therefore, in view of (3.5), there exists a unique number $\hat{r}$ satisfying $\Psi(\hat{r}) = 0$ with $\hat{r} \geq \sqrt{x_0}$, such that $\Psi(r) < 0$ in $[0,\hat{r})$ and $\Psi(r) > 0$ in $(\hat{r},\infty)$. This completes the proof of Lemma 3.2. ■

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